Completely Positive Maps on Some C*-Algebras

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Abstract: Completely positive maps on the C*-algebra of the canonical anticommutation relations induced by contractions on the underlying Hilbert space are investigated and a Stinespring decomposition exhibited. These maps are used to construct examples of dynamical semi-groups; their explicit Stinespring decompositions yield unitary dilations. The relation of this work with earlier results on the C*-algebra of the canonical commutation relations is discussed.

§1. Introduction

A *dynamical semi-group* on a C*-algebra M (with identity) is a semi-group $\{T_t : t \geq 0\}$ of completely positive linear maps of M into itself such that (i) $T_o = i_M$, (ii) $T_t(1) = 1$ for all $t \geq 0$. Examples of such semi-groups arise in quantum statistical mechanics. The above definition is due to Lindblad [6] who classified the generators of normal norm-continuous dynamical semi-groups on hyperfinite factors on separable Hilbert spaces. In [3] we proved the existence of isometric and unitary dilations of such semi-groups on the algebra of all bounded operators on a separable Hilbert space. Examples of weakly continuous dynamical semi-groups which are not norm-continuous have been provided by Davies using the Fock construction. In [4] we proved the existence of a large class of dynamical semi-groups using arbitrary representations of the CCR. In this paper we use a different method to obtain the existence of a class of dynamical semi-groups on the C*-algebra of the CAR. We begin by giving a simple proof of a theorem of Hugenholtz and Kadison [5]; this yields at the same time the existence of a class of dynamical semi-groups and of their unitary dilations. In the last section we discuss the relation of this work to our work [4] on the CCR algebra.

§2. The CAR Algebra

Let H be a complex inner product space with inner product $h, h' \mapsto \langle h, h' \rangle$ linear in the first variable. A *representation of the canonical anticommutation relations* (CAR) over H is a conjugate linear map $h \mapsto b(h)$ of H into the bounded operators on some Hilbert space $\mathfrak{H}$ such that for all h, h' in H we have

$$[b(h), b(h')]_+ = 0, \quad [b(h)^*, b(h')]_+ = \langle h, h' \rangle 1. \tag{2.1}$$

Let $\mathcal{F}(H)$ be the antisymmetric Fock space over H; that is $\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H_n^{(a)}$ where $H_n^{(a)}$, $n \geq 1$ is the Hilbert space of antisymmetric tensors of degree n with inner product given by

$$\langle x_1 \wedge \dots \wedge x_n , y_1 \wedge \dots \wedge y_n \rangle = \det(\langle x_i, y_j \rangle), \tag{2.2}$$

and $H_0^{(a)}$ is the space spanned by a single unit vector Ω called the *Fock vacuum*. The *Fock representation* of the CAR is the unique conjugate linear mapping $h \mapsto a(h)$ from H into $\mathcal{L}(\mathcal{F}(H))$ such that

$$a(h)^*(x_1 \wedge \dots \wedge x_n) = h \wedge x_1 \wedge \dots \wedge x_n ; \tag{2.3}$$

note that $a(h)\,\Omega = 0$ for all h in H.

Let $\mathcal{U}(H)$ be the C*-algebra generated by $\{a(h) : h \in H\}$; it is said to be the CAR algebra over H. As a linear space it is generated by the Wick monomials

$$a(h_1)^* \dots a(h_n)^* a(h_{n+1}) \dots a(h_{n+m}). \tag{2.4}$$

Suppose $h \mapsto a'(h)$ is another representation of CAR over H and let $\mathcal{U}'(H)$ denote the C*-algebra generated by $\{a'(h) : h \in H\}$. Then by Slawny's Theorem [8] there exists an isomorphism α of $\mathcal{U}'(H)$ onto $\mathcal{U}(H)$ such that $\alpha(a'(h)) = a(h)$ for all h in H.

Let K be a Hilbert space and let $T : H \to K$ be a contraction ($\|T\| \leq 1$). Then there exists a unique contraction $\mathcal{F}(T) : \mathcal{F}(H) \to \mathcal{F}(K)$ such that for all

$x_1, \dots, x_n$ in H we have

$$\mathcal{F}(T)(x_1 \wedge \dots \wedge x_n) = (Tx_1) \wedge \dots \wedge (Tx_n). \tag{2.}$$

It is obvious that $\mathcal{F}(T_1)\mathcal{F}(T_2) = \mathcal{F}(T_1T_2)$ and that $\mathcal{F}(1) = 1$; that is to say, $\mathcal{F}$ is a functor in the category whose objects are Hilbert spaces and whose morphisms are contractions. The category whose objects are C*-algebras with identity and whose morphisms are completely positive identity preserving linear maps has been studied by Arveson [1] and others [2]. As has been pointed out in [2], this is a good definition for although a morphism π is not necessarily an algebra homomorphism it is true that if π^{-1} exists and is a morphism then π is a *-algebra isomorphism. Throughout this paper it will be understood that each C*-algebra referred to has an identity. The question arises: does a contraction $T : H \to K$ determine in a natural way a morphism $\mathcal{U}(T) : \mathcal{U}(H) \to \mathcal{U}(K)$? An affirmative answer is given in Hugenholtz and Kadison [5]; we offer a simpler proof of their result, yielding at the same time a Stinespring decomposition [9] of $\mathcal{U}(T)$.

Theorem 1. *Let $T : H \to K$ be a contraction; then there exists a unique morphism $\mathcal{U}(T) : \mathcal{U}(H) \to \mathcal{U}(K)$ whose action on Wick monomials is given by*

$$a(h_1)^* \dots a(h_m)^* a(k_1) \dots a(k_n) \to a(Th_1)^* \dots a(Tk_n). \tag{2.}$$

Furthermore

$$\mathcal{F}(T) x\, \Omega_H = \mathcal{U}(T)(x)\, \Omega_K$$

for all x in $\mathcal{U}(H)$.

Proof: First let $T : H \to K$ be an isometry. Then

$$[a(Th)^*, a(Th')]_+ = \langle Th, Th' \rangle 1$$
$$= \langle h, h' \rangle 1 ,$$

and $[a(Th), a(Th')]_+ = 0$ for all h, h' in H. Thus $h \to a(Th)$ is a representation of the CAR; hence there is a faithful representation $\mathcal{U}(T)$ of $\mathcal{U}(H)$ on $\mathcal{U}(K)$ such that $\mathcal{U}(T)(a(h)) = a(Th)$. On the other hand, direct calculations on a total set of vectors in $\mathcal{F}(H)$ shows that the morphism $\mathcal{U}(T^*) : \mathcal{U}(K) \to \mathcal{U}(H)$ defined by $\mathcal{U}(T^*)(y) = \mathcal{F}(T)^* y\, \mathcal{F}(T)$ for all y in $\mathcal{U}(K)$ whenever T is an isometry, gives on a Wick monomial

$$\mathcal{F}(T)^* a(k_1)^* \ldots\ldots a(k_n)\, \mathcal{F}(T) = a(T^*k_1)^* \ldots\ldots a(T^*k_n) \quad .$$

Now let $T : H \to K$ be a contraction; it is well known (see Sz.-Nagy and Foias [11]) that there exists a Hilbert space L and isometries $W_1 : H \to L$, $W_2 : K \to L$ such that $T = W_2^* W_1$. Then by the above remarks $\mathcal{U}(W_2^*)$ and $\mathcal{U}(W_1)$ exist and so we may define $\mathcal{U}(T)$ by

$$\mathcal{U}(T) = \mathcal{U}(W_2^*) \circ \mathcal{U}(W_1); \tag{2.7}$$

its action on Wick monomials is seen to be given by (2.4). It follows by direct calculation that

$$\mathcal{U}(T)(x)\,\Omega_K = \mathcal{F}(T)(x\,\Omega_H)$$

for all x in $\mathcal{U}(H)$.

Remarks:

1. A Stinespring decomposition [9] of the morphism $\mathcal{U}(T)$ is given by (2.7).

2. It is obvious that $\mathcal{U}(T)\mathcal{U}(S) = \mathcal{U}(TS)$ and that $\mathcal{U}(1) = 1$ so that $\mathcal{U}$ is a functor from the category of Hilbert spaces to the category of C*-algebras.

3. Regarded as a map from $\mathcal{U}(H)$ to $\mathbb{C}$, $\mathcal{U}(0)$ is a state; it is, in fact, the Fock state $\mathcal{U}(0)(x) = \langle \Omega_H, x\,\Omega_H\rangle$ for all x in $\mathcal{U}(H)$.

4. The Fock state is invariant under $\mathcal{U}(T)$: for all x in $\mathcal{U}(H)$ we have $\langle (T)(x)\,\Omega_K, \Omega_K\rangle = \langle x\,\Omega_H, \Omega_H\rangle$. By 3, this is a consequence of 2.:

$$\mathcal{U}(0) \circ \mathcal{U}(T) = \mathcal{U}(0) \;.$$

5. Let $\mathcal{U}'(H)$, $\mathcal{U}'(K)$ be the C*-algebras generated by arbitrary representations $h \mapsto a'(h)$, $k \mapsto a'(K)$ of the CAR over H, K respectively. By Slawny's theorem there are isomorphisms $\alpha_H : \mathcal{U}'(H) \to \mathcal{U}(H)$ and $\alpha_K : \mathcal{U}'(K) \to \mathcal{U}(K)$ such that $\alpha_H(a'(h)) = a(h)$ and $\alpha_K(a'(k)) = a(k)$. Then the morphism $\mathcal{U}'(T) : \mathcal{U}'(H) \to \mathcal{U}'(K)$ defined by $\alpha_K^{-1} \circ \mathcal{U}(T) \circ \alpha_K$ has the following action on Wick monomials:

$$a'(h_1)^* \ldots\ldots a'(h_n) \longmapsto a'(Th_1)^* \ldots\ldots a'(Th_n) \;.$$

6. Let T be a fixed contraction. Suppose the representation $h \mapsto a'(h)$ of the CAR over H acts on a Hilbert space containing a cyclic vector Ω_H and the representation $k \mapsto a'(k)$ of the CAR over K acts on a Hilbert space containing a cyclic vector Ω_K, and that for all x in $\mathcal{U}'(H)$ we have

$$\langle \mathcal{U}'(T)(x)\,\Omega_K, \Omega_K\rangle = \langle x\,\Omega_H, \Omega_H\rangle \;. \tag{2.8}$$

Then there exists a unique contraction

$$\mathcal{F}'(T) : V\{x\,\Omega_H : x \in \mathcal{U}'(H)\} \to V\{y\,\Omega_K : y \in \mathcal{U}'(K)\}.$$

such that for all x in $\mathcal{U}'(H)$ we have

$$\mathcal{F}'(T)\, x\,\Omega_H = \mathcal{U}'(T)(x)\,\Omega_K \;. \tag{2.9}$$

This follows from the Schwarz inequality for completely positive maps, as in the proof of Theorem 3 of [4]. Putting $x = 1$ we have $\mathcal{F}'(T)\,\Omega_H = \Omega_K$; from (2.9) and (2.8) we have

$$\langle \mathcal{F}'(T)^* \Omega_K, x\Omega_H \rangle = \langle \Omega_H, x\Omega_H \rangle$$

for all x in $\mathcal{U}'(H)$ so that $\mathcal{F}'(T)^* \Omega_K = \Omega_H$. Thus

$$\mathcal{F}'(T)\, x\, \mathcal{F}'(T)^* \Omega_H = \mathcal{U}'(T)(x)\, \Omega_H$$

for all x in $\mathcal{U}'(H)$; in the Fock case we have $\mathcal{F}(T)^* = \mathcal{F}(T^*)$ and $\mathcal{U}(T)x\,\mathcal{F}(T)^* = \mathcal{U}(T)(x)$ for all x in $\mathcal{U}(H)$ if and only if T^* is an isometry.

7. A conjugate linear mapping $\Lambda : H \to H$ is said to be a *conjugation* on H if $\Lambda^2 = 1$ and $\langle \Lambda h, \Lambda h' \rangle = \langle h', h \rangle$ for all h, h' in H. Fix conjugations Λ_H and Λ_K in H and K respectively. Then corresponding to each contraction $T : H \to K$ there is a morphism $\overline{\mathcal{U}}(T) : \mathcal{U}(H) \to \mathcal{U}(K)$ given on anti-Wick monomials by

$$a(h_1) \,\ldots\, a(h_n)^* \longmapsto a(\overline{T}h_1) \,\ldots\, a(\overline{T}h_n)^*$$

where $\overline{T} = \Lambda_K T \Lambda_H$. Regarded as a map from $\mathcal{U}(H)$ to $\mathbb{C}$, $\overline{\mathcal{U}}(0)$ is a state which is called the anti-Fock state ($\overline{\mathcal{U}}(0)$, unlike $\overline{\mathcal{U}}(T)$ with $T \neq 0$, is independent of the particular choice of conjugations). It follows that the comment of Hugenholtz and Kadison [5] concerning gauge invariant quasi-free states on $\mathcal{U}(H)$ can be formulated as follows: a map $\phi : \mathcal{U}(H) \to \mathbb{C}$ is a gauge invariant quasi-free state with one-particle operator $A : H \to H$ $(0 \le A \le 1)$ if and only if

$$\phi = \overline{\mathcal{U}}(0) \circ \mathcal{U}(A^{\frac{1}{2}}) .$$

§3. Dynamical Semigroups

The results of the preceding section can be extended to strongly continuous one-parameter groups of contractions, yielding the existence of dynamical semi-groups on $\mathcal{U}(H)$. In [3] we discussed unitary dilations of dynamical semi-groups on von Neumann algebras [6]. To define a unitary dilation of a dynamical semi-group on a C*-algebra with identity we require the following concepts. Let $e : A \to B$ be an embedding of a C*-algebra A into a C*-algebra B such that $1_A \mapsto 1_B$. Let $n : B \to A$ be a completely positive projection of norm one such that $n \circ e = i_A$. It follows [12] that n is a conditional expectation (that is, $n(b(e \circ n)(b')) = n(b)\, n(b') = n((e \circ n)(b)\, b')$ for all b, b' in B). Now let B be a concrete C*-algebra acting on a Hilbert space $\mathfrak{H}$ and let $t \mapsto U_t$ be a strongly continuous one-parameter unitary group on $\mathfrak{H}$ such that for all $t \ge 0$

$$U_t^* B U_t \subset B .$$

We say that (e, n, U_t) is a unitary dilation of a dynamical semi-group $T_t : A \to B$ if for all $t \ge 0$ and all a in A

$$T_t(a) = n(U_t^* e(a) U_t) . \qquad (3.1)$$

We recall that, by Sz.-Nagy's Theorem [10], a strongly continuous semi-group of contractions $\{A_t : t \ge 0\}$ on a Hilbert space H has a unitary dilation in the sense that there exists a Hilbert space K, an isometry $W : H \to K$ and a strongly continuous one-parameter unitary group $\{U_t : -\infty < t < \infty\}$ such that for all $t \ge 0$ we have

$$A_t = W^* U_t W . \qquad (3.2)$$

Theorem 2. *Let* $\{A_t : t \geq 0\}$ *be a strongly continuous semigroup of contractions on a Hilbert space* H. *Let* $\mathcal{U}(H)$ *be the* CAR *algebra over* H. *Then* $t \mapsto \mathcal{U}(A_t)$ *is a dynamical semi-group with a unitary dilation*

$$\mathcal{U}(A_t) = \mathcal{U}(W^*) \circ \mathcal{U}(U_t) \circ \mathcal{U}(W) \tag{3.3}$$

where $A_t = W^* U_t W$ *is a unitary dilation. The map* $t \mapsto \mathcal{U}(A_t)(x)$ *is continuous in the weak operator topology on* $\mathcal{U}(H)$.

Proof: Since W^*, W, U_t, A_t are all contractions we have by Theorem 1, Remark 2, that

$$\mathcal{U}(A_t) = \mathcal{U}(W^* U_t W) = \mathcal{U}(W^*) \circ \mathcal{U}(U_t) \circ \mathcal{U}(W).$$

We embed $\mathcal{U}(H)$ in $\mathcal{U}(K)$ by $\mathcal{U}(W)$ and project back with $\mathcal{U}(W^*)$, $\|\mathcal{U}(W^*)\| = 1$. Moreover, by Theorem 1, Remark 6, we have

$$\mathcal{U}(U_t)(y) = \mathcal{F}(U_t)\, y\, \mathcal{F}(U_t)^*$$

so that $\mathcal{F}(U_t)^* \mathcal{U}(K) \mathcal{F}(U_t) \subset \mathcal{U}(K)$ for all t. The continuity assertion is easily checked (take x to be a Wick monomial and evaluate $\langle \mathcal{U}(A_t)(x)\, \xi, \xi \rangle$ with $\xi = \phi_1 \wedge \dots \wedge \phi_n$; then use density).

Remarks:

1. An isometry $W : H \to K$ can be factored into a unitary map $V : H \to W(H)$ together with an injection $i : W(H) \to K$. Hence $\mathcal{U}(W) = \mathcal{U}(i) \circ \mathcal{U}(V)$, but $\mathcal{U}(i)(x) = x \otimes 1$ (here 1 is the identity on $\mathcal{F}(W(H)^\perp)$ so that

$$\mathcal{U}(W)(x) = \mathcal{F}(V)\, x\, \mathcal{F}(V)^* \otimes 1 . \tag{3.4}$$

2. Suppose the A_t are isometries. Then a stronger result holds: we can choose W so that $WA_t = U_t W$ for all $t \geq 0$. We have $\mathcal{U}(W) \circ \mathcal{U}(A_t) = \mathcal{U}(U_t) \circ \mathcal{U}(W)$ so that by (3.4) there is a representation $x \to \pi(x) = \mathcal{F}(V)\, x\, \mathcal{F}(V)^* \otimes 1$ of $\mathcal{U}(H)$ such that

$$\pi(\mathcal{U}(A_t)(x)) = \mathcal{F}(U_t)\, \pi(x)\, \mathcal{F}(U_t)^* . \tag{3.5}$$

3. By the argument in Theorem 1, Remark 5, the statement of Theorem 2 holds with an arbitrary representation of the CAR in place of the Fock representation.

§4. The CCR Algebra

In [4] we showed by a different method that similar results hold for the CCR algebra over a complex Hilbert space H. Recall that *a representation of the canonical commutation relations* (CCR) over H is a map $h \mapsto W(h)$ of H into the unitary operators on some Hilbert space $\mathcal{G}$ such that

$$W(h)\, W(h') = \omega(h, h')\, W(h + h')$$

for all h, h' in H, where $\omega(h, h') = \exp\{\frac{i}{2} \operatorname{Im}\langle h,h'\rangle\}$ and such that $t \mapsto W(th)$ is strongly continuous. Let $h \mapsto W(h)$ be a representation of the CCR over H and let $\mathcal{W}(H)$ be the C*-algebra generated by the Weyl operators $\{W(h) : h \in H\}$.

A function $\mu : H \to \mathbb{C}$ is said to be a *generating functional* if (i) $\mu(0) = 1$, (ii) $h, h' \mapsto \omega(h,h')\, \mu(h - h')$ is positive definite, (iii) for each h, h' in H the map $t \mapsto \mu(h + th')$ is continuous. Let $T : H \to K$ be a bounded operator. Choose a generating functional μ on each Hilbert space; then μ is said to be T-positive if (i) $\mu(Th) \neq 0$ for all h in H, (ii) the kernel

$$h, h' \longmapsto \frac{\mu(h - h')\, \omega(h, h')}{\mu(T(h - h'))\, \omega(Th, Th')}$$

is positive definite. In [4] we proved that if $h \mapsto W(h)$ is a representation of the CCR over H, if $\mathcal{W}(H)$ is the C*-algebra which it generates, and if μ is a T-positive generating functional on H then there exists a morphism $\mathcal{W}(T)$ of $\mathcal{W}(H)$ into $\mathcal{W}(K)$ such that

$$W(h) \longmapsto W(Th)\, \frac{\mu(h)}{\mu(Th)} .$$

In particular if μ is T-positive whenever $T : H \to K$ is a contraction then $\mathcal{W}$ is a functor. We pointed out that this occurs when μ has the same form $\mu(h) = e^{-\frac{\lambda}{4}\|h\|^2}$ for each Hilbert space and $\lambda \geq 1$ is fixed (these are the

extremal universally invariant generating functionals introduced by Segal [7] which include the Fock case (λ = 1)). The Fock case can be dealt with by the methods of this paper, and we will now sketch the proof.

Suppose first that $T : H \to K$ is an isometry and let $\mathcal{W}(H)$ and $\mathcal{W}(K)$ be the C*-algebras generated by the Fock representations of the CCR over H and K respectively. Then there is a homomorphism $\mathcal{W}(T) : \mathcal{W}(H) \to \mathcal{W}(K)$, since T preserves the multiplier ω. Furthermore there is a morphism $\mathcal{U}(T^*) : \mathcal{U}(K) \to \mathcal{U}($ given by

$$\mathcal{U}(T^*)(y) = \mathcal{F}(T^*)(y)\, \mathcal{F}(T^*)^* .$$

By direct calculation on vectors of the form $W(h')\,\Omega_H$ we find that

$$\mathcal{U}(T^*)(W(h)) = W(T^*h)\, e^{-\frac{1}{4}\{\|h\|^2 - \|T^*h\|^2\}} . \tag{4.1}$$

Now suppose that $T : H \to K$ is a contraction and put $T = W_2^* W_1$ where $W_1 : H \to L$ and $W_2 : K \to L$ are isometries. Then define $\mathcal{W}(T)$ by $\mathcal{W}(T) = \mathcal{W}(W_2^*) \circ \mathcal{W}(W_1)$. Then check that on W(h) we have

$$\mathcal{W}(T)(W(h)) = W(Th)\, e^{-\frac{1}{4}\{\|h\|^2 - \|Th\|^2\}} .$$

The Fock generating functional entered at (4.1). The relation with §2 can be seen formally as follows: we have

$$W(h) = e^{\frac{i}{\sqrt{2}} a(h)^*}\, e^{\frac{i}{\sqrt{2}} a(h)}\, e^{-\frac{1}{4}\|h\|^2} ;$$

the right hand side is a sum of Wick monomials, and applying the rule of Theorem 1 to them we would obtain

$$W(h) \longmapsto W(Th)\, \frac{e^{-\frac{1}{4}\|h\|^2}}{e^{-\frac{1}{4}\|Th\|^2}} .$$

The result may be transferred to an arbitrary representation of the CCR using the Slawny isomorphism as in §2 but the multiplying factor still remains the one got from the Fock generating functional.

We remark here that if $\mathcal{W}_\lambda$ is the functor got from the generating functional

$$\mu_\lambda(h) = e^{\frac{-\lambda}{4}\|h\|^2}$$

and $t \mapsto A_t$ is a strongly continuous semi-group of contractions on H then a unitary dilation of the dynamical semi-group $t \mapsto \mathcal{W}_\lambda(A_t)$ is obtained as in §3 using a unitary dilation $A_t = W^*U_tW$ of A_t:

$$\mathcal{W}_\lambda(A_t) = \mathcal{W}_\lambda(W^*) \circ \mathcal{W}_\lambda(U_t) \circ \mathcal{W}_\lambda(W) .$$

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References

[1] Arveson, W. B.: Subalgebras of C*-algebras. Acta Math. 123, 141 - 224 (1969).

[2] Choi, M. D.: Completely positive linear maps on complex matrices. Linear Algebra and its Applications 10, 285 - 290 (1975).

Effros, E. G., Lance, E. C.: Tensor products of operator algebras. Advances in Maths. (to appear).

Evans, D. E.: Completely positive bounded linear maps. Preprint DIAS-TP-75-45.

[3] Evans, D. E., Lewis, J. T.: Dilations of dynamical semi-groups. Preprint DIAS-TP-76-4.

[4] Evans, D. E., Lewis, J. T.: Completely positive maps on the CCR algebra. Preprint DIAS-TP-76-14.

[5] Hugenholtz, N. M., Kadison, R. V.: Automorphisms and Quasi-free States on the CAR algebra. Commun. Math. Phys. 43, 181 - 197 (1975).

[6] Lindblad, G.: On the generators of quantum dynamical semi-groups. Preprint, Royal Institute of Technology, Stokholm. TRITA-TFY-75-1.

[7] Segal, I. E.: Mathematical characterization of the physical vacuum for a linear Bose-Einstein field. Illinois J. Math. 6, 500 - 523 (1962).

[8] Slawny, J.: On factor representations and the C*-algebra of canonical commutation relations. Commun. Math. Phys. 24, 151 - 170 (1972).

[9] Stinespring, W. F.: Positive functions on C*-algebras. Proc. Amer. Math. Soc. 6, 211 - 216 (1955).

[10] Sz.-Nagy, B.: Unitary dilations of semi-groups of contractions. Acta Scientiarum Math. Szeged 15, 104 - 114 (1954).

[11] Sz.-Nagy, B., Foias, C.: Harmonic Analysis of Operators on Hilbert Space. Amsterdam: North-Holland, 1970.

[12] Tomiyama, J.: Tensor products and projections of norm one in von Neumann algebras. Seminar report, University of Copenhagen, 1970.